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Hydraulic control by a wide weir in a rotating fluid

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Flow control by a wide, deep weir in a rotating fluid is investigated theoretically and experimentally. A strong (vertical) vorticity constraint due to frame rotation is combined with conservation of the Bernoulli function along streamlines and a standard hydraulic control assumption to show that the volume flux over the barrier is

$$Q = g^{-1}[\tfrac{2}{3}g(H-b_0) - \tfrac{1}{3}f^2l^2]^{\frac{3}{2}},$$

where H is the depth of the fluid column upstream, b_0 is the crest height, f is the Coriolis parameter, and l is a length-scale measure of the breadth of the weir. The component of the velocity parallel to the weir crest is computed from conservation of potential vorticity to be $v = -fl$; perpendicular to the crest, we recover the standard hydraulic relation $u = (gh_0)^{\frac{1}{2}}$.

Experimental investigations of upstream height and streamline deflexion as functions of rotation are described. It is found that agreement with theory is good up to a certain rate of rotation, above which the finite width of the experimental weir becomes important.

1. Introduction

Some of the oldest and most widely used solutions to the equations for inviscid nonlinear flow have been derived for the case of fluid motions over large barriers such as weirs. The simplest example, that of homogeneous fluid flowing over a deep, 'broad-crested' weir in an inertial frame of reference, was developed in the 18th century; much more recent work by Long (1954) extended the analysis to include a density stratification in the fluid. Another obvious extension of the classical hydraulic solutions – that of inertial flows in a rotating fluid – has only recently been examined by Whitehead, Leetmaa & Knox (1974) for the special case of flow through a long, straight channel. In this paper we present theoretical and experimental evidence that the classical problem of flow over a 'broad-crested' weir can be extended to include the effects of a strong frame rotation. It is shown that a steady-state solution can be obtained for inviscid

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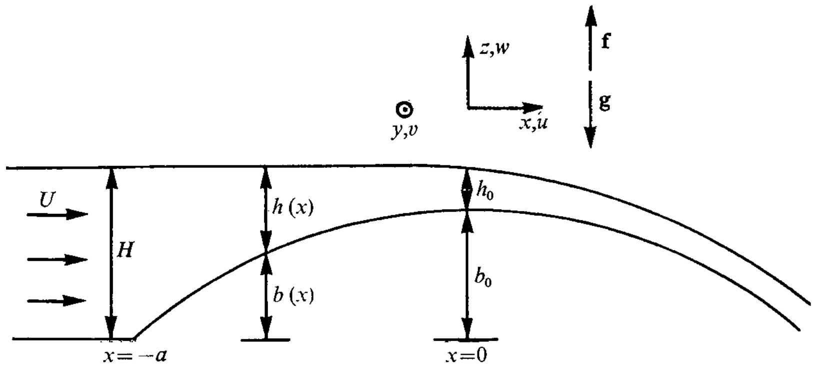

FIGURE 1. A sketch of the geometry. The notation is explained in the text.

nonlinear flow over a large barrier in a rotating fluid, subject to the usual depth-averaging assumption.

There are some obvious phenomena in geophysics and engineering where such rotating, nonlinear flow regimes may be important. One thinks immediately of water flow through oceanic sills, or airflow over long mountain ranges in the atmosphere. Perhaps there are applications in civil or hydraulic engineering. There may even be analogues in rotating fluids for the peculiar effects introduced into these hydraulics problems by stratification, such as upstream blocking, etc. But these are left as topics for future study. For the time being, we restrict ourselves to the presentation of a simple theoretical and experimental analysis of fully nonlinear flow over a barrier in a rotating fluid.

2. Theoretical development

We consider the simplified geometry of figure 1. The fluid upstream flows in a column of thickness H with speed U. The barrier, which extends infinitely in the y direction, has its base at $x = -a$, and the crest (of height b_0) is at $x = 0$. $z = b(x)$ describes the form of the obstacle, $h(x)$ is the thickness of the fluid column, and $z = b(x)+h(x)$ denotes the free surface. The rotation vector $\mathbf{\Omega} = \frac{1}{2}\mathbf{f}$ and the gravity vector $-\mathbf{g}$ lie in the z direction.

We start with the frictionless Navier–Stokes equations in a steady state:

$$\nabla . \mathbf{u} = 0, \tag{2.1}$$

$$\mathbf{u}.\nabla\mathbf{u}+f\hat{\mathbf{k}}\times\mathbf{u} = -\rho^{-1}\nabla p, \tag{2.2}$$

where $\mathbf{u}$ is the velocity vector, $f\hat{\mathbf{k}}$ is twice the (vertical) frame rotation, ρ is density and p is pressure.

The well-known inertial–rotational equations, used for instance in Charney's (1955) inertial Gulf Stream theory, can be derived from the above equations by assuming that the ratio of total depth of the fluid H is very much smaller than a typical horizontal scale of the variation of the boundaries, L. In the limit

$H/L \ll 1$, it is well known (see Stern 1975, pp. 31–33) that the potential vorticity is conserved, i.e.

$$(\mathbf{u}.\nabla)\left[\frac{f+\omega}{h(x)}\right] = 0, \tag{2.3}$$

where ω is vertical vorticity of the fluid. The Bernoulli function

$$G = g[h(x)+b(x)]+\tfrac{1}{2}(\mathbf{u}.\mathbf{u})$$

is conserved along streamlines, with cross-stream variations in G being determined by the vorticity constraint. Likewise, from (2.3) we can see that the potential vorticity

$$F = (f+\omega)/h(x)$$

is conserved along a streamline. Obviously, F and G are not independent; it can be shown that for a mass-transport stream function ψ, defined by

$$\nabla \times \psi\mathbf{k} = h\mathbf{u},$$

the relationship $dG/d\psi = F$ must hold (see Charney 1955). Thus, in order to obtain a consistent solution, we must specialize upstream conditions so that $dG/d\psi = F$ everywhere. To do this we shall make the weir very deep, so that H is large with respect to the height at the crest h_0, but we shall assume that L is even larger, so that $H/L \ll 1$. The former assumption enables us to set $F = f/H$, which we shall assume is close to zero so that G becomes a constant, which we shall set equal to gH.

Since there are no variations across the weir, (2.3) becomes

$$\frac{f+dv/dx}{h(x)} = \frac{f+[dv/dx]_{-a}}{H}. \tag{2.3a}$$

To integrate (2.3*a*) it is convenient to introduce the length scale

$$l(x) = -\int_{-a}^{x}\left[\frac{h(x')}{H}\left(1+\frac{1}{f}\frac{dv}{dx}\bigg|_{-a}\right)-1\right]dx'. \tag{2.4}$$

There are two particularly interesting upstream cases. The first and simplest results when there is absolutely no cross-stream tilt so that $[dv/dx]_{-a} = -f$ and hence $l = x+a$. The second results when there is a geostrophic balance with an accompanying but small cross-stream tilt, so that $[dv/dx]_{-a} = 0$. Using $H \sim b_0$, l becomes

$$l \sim \frac{1}{b_0}\int_{-a}^{x} b(x')\,dx'.$$

The last integral is always less than $x+a$.

The cross-stream velocity thus integrates to

$$v = -fl(x). \tag{2.5}$$

Substitution of the expression (2.5) into Bernoulli's equation gives

$$u^2+f^2l^2+2g(h+b) = 2gH, \tag{2.6}$$

or

$$u(x) = [2g(H-h(x)-b(x))-f^2l^2(x)]^{\frac{1}{2}}, \tag{2.6a}$$

and the volume flux is

$$Q = uh = h(x)\,[2g(H-h(x)-b(x))-f^2l^2(x)]^{\frac{1}{2}}. \tag{2.7}$$

To close the problem, we need a further relationship to connect the variables u and h. If the barrier is to act as a hydraulic control, the results of non-rotating hydraulics tell us that there is an explicit relationship among Q, u and h such that if one of these three quantities is known, the other two are uniquely determined (the critical condition). Several such relationships have been developed; all have been shown to be equivalent to each other in non-rotating hydraulics. The most convenient relation for use here is that

$$\partial Q/\partial h = 0 \quad \text{at the crest.} \tag{2.8}$$

We apply this criterion to (2.7) to obtain

$$h_0 = \tfrac{2}{3}(H-b_0)-\tfrac{1}{3}g^{-1}f^2l^2 \tag{2.9}$$

and

$$Q = g^{-1}[\tfrac{2}{3}g(H-b_0)-\tfrac{1}{3}f^2l^2]^{\frac{3}{2}}. \tag{2.10}$$

Frame rotation thus acts as a block to steady flow: for fixed volume flux Q, H must increase as f is increased if a steady state is to be maintained. Note that when $f = 0$ we recover the classical formula for steady discharge per unit width over a broad-crested weir.

It has been pointed out by a referee that if (2.7) is written as

$$\frac{Q^2}{2gh(x)^2}+h(x) = H-\tfrac{1}{2}g^{-1}f^2l^2(x)-b(x)$$

then the 'control point' occurs at that value of x for which

$$\frac{d}{dx}[b(x)+\tfrac{1}{2}g^{-1}f^2l^2(x)] = 0,$$

and therefore the control point will shift from the place where $db/dx = 0$ as rotation increases. Using the approximation that $dl/dx \simeq 1$, the control point will exist at $db/dx = -f^2l/2g$. The parameter group $\frac{1}{2}f^2l/g$ is always smaller than $(H-b_0)/l$ by virtue of (2.10), and $(H-b_0)/l$ is a small number for the approximation considered here where $H/L \ll 1$. It may be possible to conduct experiments to observe this feature but it has not been done here.

3. Experimental evidence

The theoretical predictions deduced in §2 were tested experimentally in the tank system shown in figure 2 (plate 1). The tank measures 90 cm long by 25·7 cm wide by 50 cm deep. Built onto the bottom is a large paraboloidal barrier, 60 cm long with an apex 16 cm above the base. A submersible pump downstream recirculates the fluid to an upstream diffusion system of horsehair fibre and a sprinkler encased in a packed gravel rockbed.

In experimental runs the entire tank system was mounted on a variable-speed rotating turntable. Two measurements will be described here: (i) upstream fluid

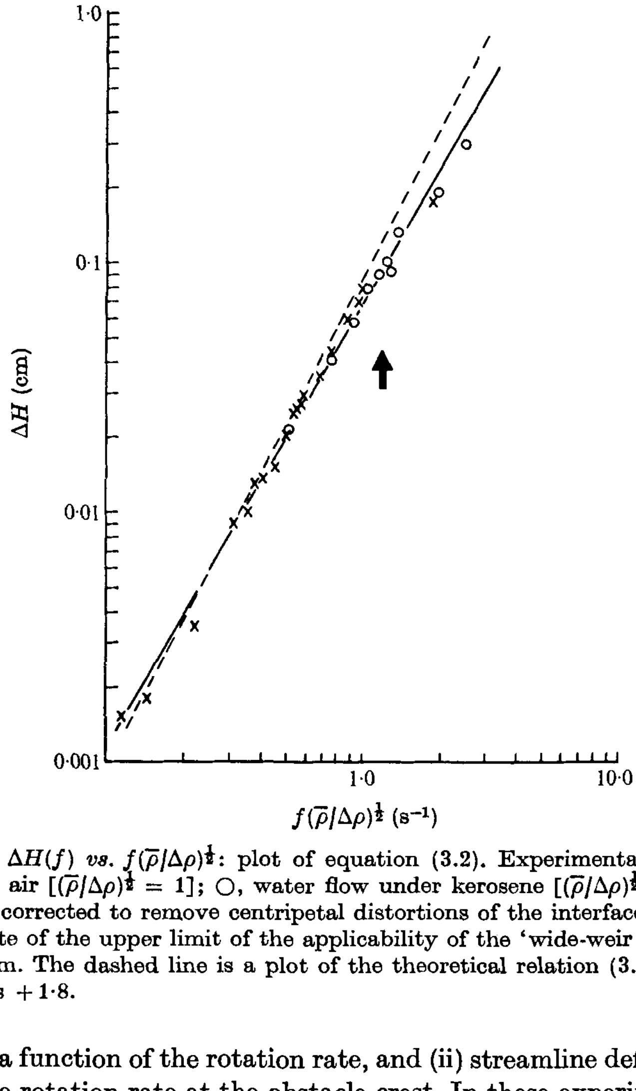

FIGURE 3. $\Delta H(f)$ *vs.* $f(\bar{\rho}/\Delta\rho)^{\frac{1}{2}}$: plot of equation (3.2). Experimental points: ×, water flow under air $[(\bar{\rho}/\Delta\rho)^{\frac{1}{2}} = 1]$; ○, water flow under kerosene $[(\bar{\rho}/\Delta\rho)^{\frac{1}{2}} = 2{\cdot}16]$. The data have been corrected to remove centripetal distortions of the interface. The arrow marks our estimate of the upper limit of the applicability of the 'wide-weir' assumption to our tank system. The dashed line is a plot of the theoretical relation (3.2); the slope of the solid line is $+1{\cdot}8$.

height as a function of the rotation rate, and (ii) streamline deflexions as a function of the rotation rate at the obstacle crest. In these experiments, the 'wide-weir' assumption is valid only for sufficiently low rotations; if the rotation rate is high enough, trajectories will be so curved that fluid parcels will reach the rotation-lagging side wall before they reach the crest. In that case, the volume flux over the barrier is the result of a narrow, fast current pushed up against the rotation-lagging side of the tank. This transition to another hydraulic-control regime for higher rotation rates is evident in the experimental data. For water under air, the transition occurs at around $f = 1{\cdot}2\ \mathrm{s}^{-1}$, and less for water under kerosene.

3.1. *Upstream height measurements*

If, in (2.10), Q is fixed and f varies, then in order to maintain the critical steady state, we must have

$$H(f) = H(f = 0) + \tfrac{1}{2}g^{-1}f^2l^2, \qquad (3.1)$$

where $H(f)$ is the upstream thickness of the fluid column (not including centripetal distortions of the free surface) and $l = l(0)$, or the length-scale measure of the weir breadth. Define

$$\Delta H(f) = H(f) - H(0).$$

Then

$$\Delta H(f) = \tfrac{1}{2}g^{-1}f^2l^2. \tag{3.2}$$

Equation (3.2) can be tested directly through micrometer readings of the free-surface heights for various values of f with the pump discharge held constant. In the experiments, the micrometer's position was carefully noted so that corrections for the centripetal distortions of the free surface could be made. The data presented here have been corrected for the centripetal distortions.

Figure 3 presents the results of several experimental runs: some that measure free-surface elevations of water under air ($g\Delta\rho/\bar{\rho} = g = 980$ cm s^{-2}) and others that measure interface elevations of water under kerosene ($g\Delta\rho/\bar{\rho} = g' = 210$ cm s^{-2}). The independent variable f has been replaced by the quantity $f(\bar{\rho}/\Delta\rho)^{\frac{1}{2}}$ so that all data could be plotted together. In the experiments, $l = l(0) = 12{\cdot}7$ cm, using the second definition of l.

It is easily seen by inspection of figure 3 that a pronounced increase in the upstream height is necessary to maintain the critical steady-state flow over the weir in the presence of frame rotation. Upstream height values for the water–air interface fall on an f^2 line, and for higher values of $f(\bar{\rho}/\Delta\rho)^{\frac{1}{2}}$ (water–kerosene interface), with fluid-parcel trajectories more curved and the wide-weir assumption less valid, the data points fit an $f^{1{\cdot}8}$ line better. However, even in the rapidly rotating limit, the height dependence on the rotation rate is much more pronounced than would be expected from a cross-stream geostrophic balance like that formulated by Whitehead *et al.* (1974). In their rapidly rotating limit, $\Delta H \propto f^{\frac{1}{2}}$. Thus we can assume that the flow regime in the laboratory model is always highly nonlinear.

3.2. *Streamline deflexions*

Streamline deflexions can be easily calculated: we have

$$ds/dx = \tan\theta = v/u, \tag{3.3}$$

where $y = s(x)$ is the equation for streamline paths. At the crest, this becomes

$$\frac{ds}{dx} = \frac{-fl}{[\frac{2}{3}g(H-b_0) - f^2l^2]^{\frac{1}{2}}}. \tag{3.4}$$

If

$$\frac{f^2l^2}{\frac{2}{3}g(H-b_0)} \ll 1,$$

which holds in the case of the experiments reported here,

$$(ds/dx)_{\text{crest}} \simeq -fl[\tfrac{2}{3}g(H-b_0)]^{-\frac{1}{2}}, \tag{3.5}$$

or

$$\tan\theta_c \propto f \qquad [\tan\theta_c = (ds/dx)_{\text{crest}}].$$

Thus the quantity $f^{-1}\tan\theta_c$ should be a constant.

The angle θ was measured by photographing the angle of four long wooden floats attached to small threads which held them over the crest of the weir.

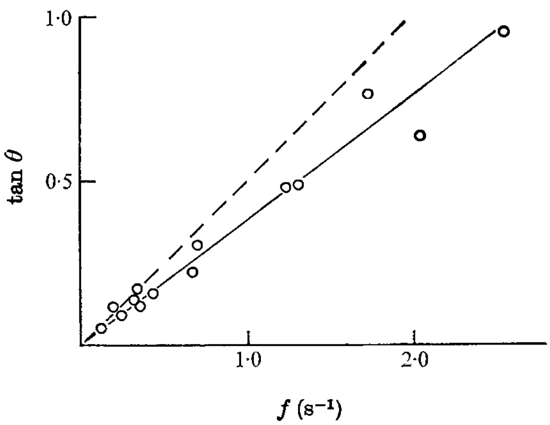

FIGURE 5. tan θ_c *vs*. f. θ_c is the angular streamline deflexion from a line perpendicular to the crest. The lower line (slope 0·38) is a best fit to the data; the upper line (slope 0·5) gives the slope from equation (3.5).

Figure 4 (plate 1) shows deflexion of the floats at progressively greater rates of rotation, the rates being $f = 0$, 0·35, 0·84, 1·29 and 2·05 s^{-1}. Since $\Delta\rho/\bar{\rho} = 1$, only the first three photographs correspond to the limit in which the wide-weir approximation can be expected to be valid. Figure 5 shows measurements of the angles as a function of f as compared with the prediction (3.5). There is reasonable agreement, especially in the lower range of f. However, the slope of the data-fit line ($\sim$ 0·38) is lower than the theoretical value, which is 0·5. We attribute this to the effects of finite tank width.

4. Summary and conclusions

It has been shown that a solution for steady fluid disoharge over a wide, broad-crested weir in a rotating frame does exist. The theoretical predictions have been corroborated by laboratory experiments.

To obtain a solution, it has been necessary to make the assumptions that the upstream fluid is very deep and stagnant, and that the pressure distribution is everywhere hydrostatic. If the fluid were not deep or the pressure not hydrostatic, a solution driven solely by an upstream potential head could not be derived: the vorticity constraint could not be accommodated. Of course, forced-flow solutions with an imposed upstream Froude number and cross-stream geostrophic balance would constitute an interesting extension of this work.

It is possible that nonlinear rotating flows exist in nature. Previous studies of hydraulic applications in geophysics (especially meteorology) neglected Coriolis accelerations (Long 1954; Houghton & Kasahara 1968). Such a simplification is valid in some cases; however, it is entirely plausible that the flow regime through an ocean sill or over a mountain range adjusts itself so that steady-state discharges in the presence of strong vorticity constraints may be maintained. Such a nonlinear rotating flow is especially likely in cases where density imbalances are the principal driving force; i.e. where there is little kinetic energy (low speeds) in the upstream basin. The theory developed here indicates that frame rotation would lead to significant differences in flux over the weir when the parameter group $\rho f^2 l^2/2g\Delta\rho(H-b_0)$ is of order one or larger. This condition may be satisfied occasionally for atmospheric flows over long, high mountain ranges, the Andes being the best example of such a barrier in the atmosphere. This condition is certainly satisfied in the ocean, but it is not clear whether any oceanic obstacle is wide enough to satisfy the two-dimensional assumption.

We close by cautioning that since this flow does not occur in an inertial frame, towed-obstacle experiments similar to those conducted by Long (1954) are *not* equivalent to the experiments conducted above, and the theory would have to be reformulated.

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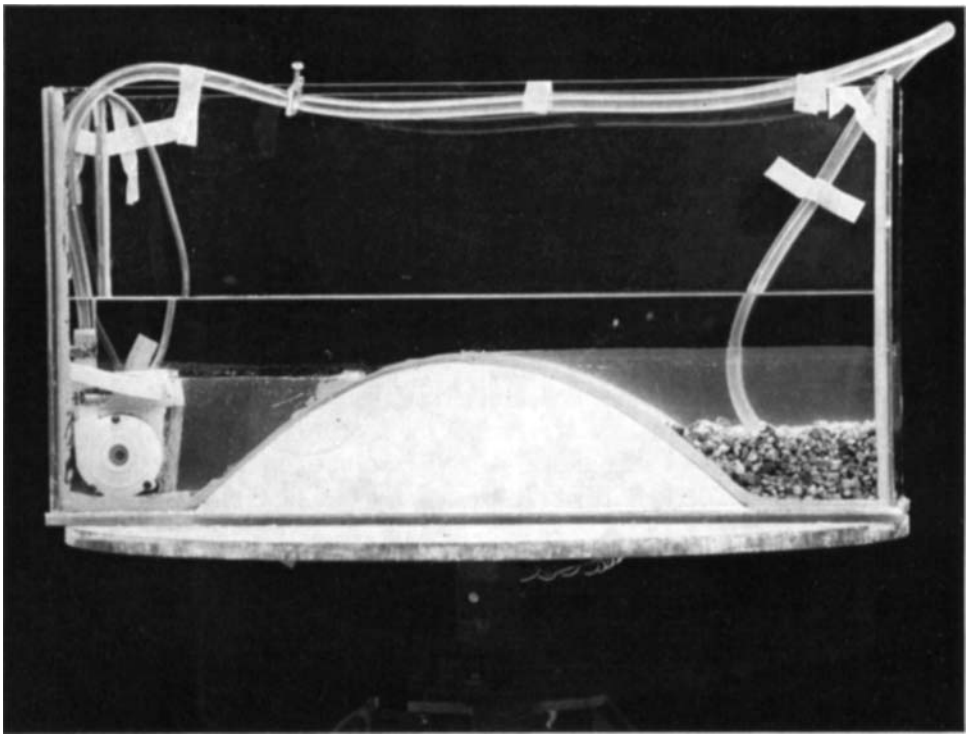

FIGURE 2. A photograph of the experimental apparatus. The lower fluid is water; the upper layer is kerosene.

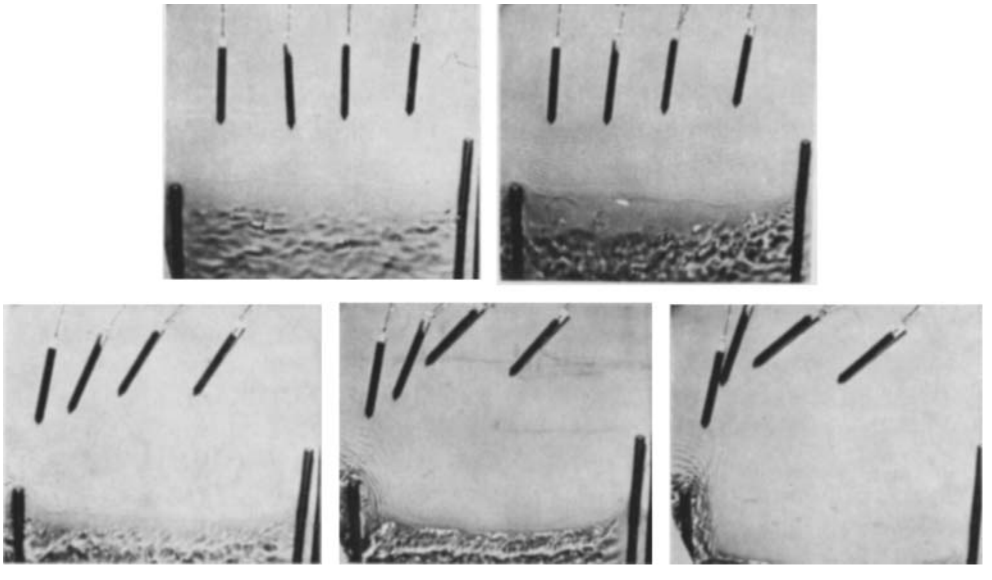

FIGURE 4. Visualization of streamline deflexions for $f = 0$, 0·35, 0·84, 1·29 and 2·05 s^{-1} at a water–air interface. The viewer is looking down from straight above the crest; water flows from top to bottom. A hydraulic jump is evident at the bottom of the photograph. Floats are located approximately at the crest of the weir.